CECS-PHY-99/14
gr-qc/9909015

Conserved charges for gravity with locally AdS asymptotics

Rodrigo Aros[1,4], Mauricio Contreras[1,4], Rodrigo Olea[1,3], Ricardo Troncoso[1,2] and Jorge Zanelli[1,2]
[1] *Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago, Chile.*
[2] *Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile.*
[3] *Departamento de Física, FCFM, Universidad de Chile, Casilla 487-3, Santiago, Chile.*
[4] *Universidad Nacional Andrés Bello, Sazie 2320, Santiago, Chile.*

Abstract. A new formula for the conserved charges in 3+1 gravity for spacetimes with local AdS asymptotic geometry is proposed. It is shown that requiring the action to have an extremum for this class of asymptotia sets the boundary term that must be added to the Lagrangian as the Euler density with a fixed weight factor. The resulting action gives rise to the mass and angular momentum as Noether charges associated to the asymptotic Killing vectors without requiring specification of a reference background in order to have a convergent expression. A consequence of this definition is that any negative constant curvature spacetime has vanishing Noether charges. These results remain valid in the limit of vanishing cosmological constant.

Noether theorem is the time honored way of constructing conserved charges for any physical system provided its symmetries have been identified. Nevertheless, it is puzzling that the straightforward application of Noether's theorem to gravitation theory does not seem to lead to the expected expressions for mass and angular momentum as the conserved charges associated to time translations an spatial rotations. Applying Noether's theorem for diffeomorphisms to the Einstein-Hilbert action in the second order formalism yields an expression for the conserved charges as a surface integral first proposed by Komar [1], (see, e.g. [2]). This expression, however, requires different normalization factors for the mass and the angular momentum in order to reproduce the weak field approximation. The generalization of Komar's formula when a negative cosmological constant is introduced, requires the further substraction of a background configuration in order to render a finite result.

The search for a general formula which allows to compute the mass (and other conserved charges) in General Relativity as a surface integral has proceeded through a series of improvements.

In the Hamiltonian framework, Arnowitt, Deser and Misner found a conserved charge which yields the right normalization [3]. As a follow up in this scheme, Abott and Deser generalized this result in case of negative cosmological constant [4].

An expression which extends and formally justifies this treatment was given by Regge and Teitelboim who showed that the conserved charge is the boundary term that must be added to the Hamiltonian constraint so that its variation be well-defined [5]. Hawking and Horowitz observed that this boundary term in the Hamiltonian can be obtained from an action principle appropriate for Dirichlet boundary conditions for the metric [6]. This action, however, depends explicitly on the background solution.

A common feature of these approaches is the need of choosing a reference background with suitable matching conditions. These conditions depend on the asymptotic and topological properties corresponding to the class of metrics involved. However, such a choice is not necessarily unique, and in some cases it might even be impossible to find one.

As it occurs in Yang-Mills theories, there are cases of physical interest where one knows the asymptotic behavior of the curvature, but has no information about the metric at infinity. In those instances, there is no unique background substraction that provides a general expression for the conserved charges which works properly for all solutions with the same asymptotic curvature. In particular, background-dependent methods are inappropriate to deal with every kind of asymptotically locally AdS (**ALAdS**) solutions.

One way to deal with this problem was proposed by Ashtekar and Magnon [7], who use conformal techniques to construct an expression for the 'conserved' quantities in asymptotically AdS spaces. The results correspond to those obtained by Hamiltonian methods [8]. In this construction there is no mention of an action, thus in principle, it is not straightforward how to relate these conserved quantities and those associated with the symmetries of an action in the Lagrangian formalism.

Another way, inspired by the AdS/CFT conjecture [9] has been recently proposed [10]. This proposal considers the Einstein-Hilbert action with a negative cosmological constant plus the Gibbons-Hawking boundary term – needed so that the variational principle be well defined for metrics satisfying Dirichlet boundary conditions–, plus an additional counterterm action which depends on the induced metric at the boundary. Thus, using the Brown-York quasilocal energy definition [11] the sought formula is obtained. However, there is no clear a priori insight as to which counterterm must be added. Any functional of the boundary metric can be used as counterterm action (see, e.g., [12,13]). The only natural criteria seem to be

requiring covariance and convergence of the expression. The result is a series which needs to be corrected as one increases the complexity of the solution.

In what follows it is shown that using the first order formalism allows to write down an action principle which has an extremum for any ALAdS solution. A regularized, background-independent expression for the conserved charges is obtained by a straightforward application of Noether's theorem.

Consider the Einstein-Hilbert action for gravity with negative cosmological constant in 3+1 dimensions. In the first order formalism, where the vierbein and the spin connection are varied independently, the action reads

$$I = \frac{\kappa}{2}\int_M \epsilon_{abcd}(R^{ab}e^c e^d + \frac{l^{-2}}{2}e^a e^b e^c e^d) + B, \qquad (1)$$

where $\kappa = 1/(16\pi G)$ and the cosmological constant is $\Lambda = -\frac{3}{l^2}$. Here B denotes a boundary term, which should be a functional of the fields at infinity. The Noether currents and charges are sensitive to this boundary term, and B can be fixed by requiring that I_G have an extremum on geometries which are ALAdS. Varying the action yields

$$\delta I = \kappa\int_M \epsilon_{abcd}[\bar{R}^{ab}e^c\delta e^d - T^a e^b \delta\omega^{cd}] + \int_{\partial M}\Theta \qquad (2)$$

where the boundary term in (2) is

$$\int_{\partial M}\Theta = \frac{\kappa}{2}\int_{\partial M}\epsilon_{abcd}e^a e^b\delta\omega^{cd} + \delta B, \qquad (3)$$

$T^a = De^a$ is the torsion two-form and $\bar{R}^{ab} := R^{ab} + l^{-2}e^a e^b$. The first integral in (2) is zero on shell and therefore, the action has an extremum, provided Θ vanishes on the classical solution. Assuming that the spacetime is ALAdS ($\bar{R}^{ab} = 0$ at ∂M), this condition is satisfied by

$$\delta B = \frac{\kappa l^2}{2}\int_{\partial M}\epsilon_{abcd}R^{ab}\delta\omega^{cd}. \qquad (4)$$

The r.h.s. of (4) can be easily recognized as $\frac{\kappa l^2}{4}$ times the variation of the Euler (Gauss-Bonnet) density,

$$\delta\left[\int_M \epsilon_{abcd}R^{ab}R^{cd}\right] = 2\int_M d\left[\epsilon_{abcd}R^{ab}\delta\omega^{cd}\right].$$

Thus, the action is fixed up to a constant as

$$I = \frac{\kappa l^2}{4}\int_M \epsilon_{abcd}\bar{R}^{ab}\bar{R}^{cd}. \qquad (5)$$

Either by direct variation of (5) or by substitution of (4) into (3), the term Θ is given by

$$\Theta(\omega^{ab}, \delta\omega^{ab}, e^a) = \frac{\kappa l^2}{2}\epsilon_{abcd}\delta\omega^{ab}\bar{R}^{ab}, \qquad (6)$$

up to an exact form, which is annihilated by the requirement $\Theta = 0$ on ∂M. On the other hand, this requirement, in turn, could be satisfied either by a Dirichlet condition on the spin connection ($\delta\omega^{ab} = 0$) or by a sort of Neumann condition which imposes the boundary of the spacetime to have negative constant curvature ($\bar{R}^{ab} = 0$). If the only boundary is the asymptotic region, this last condition is equivalent to require that spacetime be ALAdS.

The action (5) is manifestly invariant under diffeomorphisms, and therefore, Noether's theorem provides a conserved current associated with this invariance, given by

$$*J = -\Theta(\omega^{ab}, e^a, \delta\omega^{ab}) - I_\xi L, \qquad (7)$$

where the Lagrangian L and Θ are defined in (5) and in (6) respectively and I_ξ stands for the contraction operator that lowers the degree of a p-form by one [14]. For a diffeomorphism, $\delta\omega^{ab} = -\mathcal{L}_\xi\omega^{ab}$, and by virtue of (7), the conserved current can be locally written as

$$*J = \frac{\kappa l^2}{2}d(\epsilon_{abcd}I_\xi\omega^{ab}\bar{R}^{cd}), \qquad (8)$$

then, the conserved charge $Q(\xi) = \int_\Sigma *J$ is expressed as a surface integral. The final expression is

$$Q(\xi) = \frac{\kappa l^2}{2}\int_{\partial\Sigma}\epsilon_{abcd}I_\xi\omega^{ab}\bar{R}^{cd} \qquad (9)$$

where Σ a spatial section of the manifold.

In sum, the condition that the action should have an extremum for any ALAdS spacetime requires the addition of the Euler density as the boundary term with the fixed weight $-\frac{3\kappa}{4\Lambda}$. This prescription provides a simple and useful tool to compute charges in General Relativity. For instance, the energy is obtained when ξ is a time-like Killing vector.

Although the ALAdS condition implies $\bar{R}^{cd} = 0$ at the boundary, the integrand of (9) approaches a finite value on $\partial\Sigma$. In fact, standard boundary conditions [8] guarantee the convergence of the charge for spatially localized configurations. Under those same conditions, the contribution to the action from spatial infinity can be directly shown to be finite as well. In the following examples the convergence of (9) will be explicitly tested for geometries of different topologies with timelike and spacelike Killing vectors.

- **Kerr-AdS**

The Kerr-AdS geometry in Boyer-Lindquist-type coordinates [15] can be expressed choosing the local Lorentz frame

$$e^0 = \tfrac{\sqrt{\Delta_r}}{\Xi\rho}(dt - a\sin^2\theta d\varphi),\ \ e^1 = \rho\tfrac{dr}{\sqrt{\Delta_r}}\ ,$$

$$e^2 = \rho\tfrac{d\theta}{\sqrt{\Delta_\theta}},\ \ e^3 = \tfrac{\sqrt{\Delta_\theta}}{\Xi\rho}\sin\theta(adt - (r^2 + a^2)d\varphi)\ , \qquad (10)$$

with $\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr$, $\Delta_\theta = 1-\frac{a^2}{l^2}\cos^2\theta$, $\Xi = 1-\frac{a^2}{l^2}$ and $\rho^2 = r^2+a^2\cos^2\theta$. The mass and the angular momentum are found evaluating the charge (9) for the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\varphi}$, respectively

$$Q(\tfrac{\partial}{\partial t}) = \tfrac{m}{\Xi^2};\ Q(\tfrac{\partial}{\partial\varphi}) = \tfrac{ma}{\Xi^2}\ , \tag{11}$$

in agreement with Ref. [8].

Note that formula (9) yields the right values for the mass and angular momentum without need for adjusting the normalization factors as it happens using the Komar potential.

- **(Un)wrapped black string**

Another kind of ALAdS spacetime, but with different topology and different asymptotic behavior is given by the following black string geometry (see, e.g. [16,17])

$$ds^2 = -(\frac{r^2}{l^2}-\frac{b}{r})dt^2 + \frac{dr^2}{(\frac{r^2}{l^2}-\frac{b}{r})} + r^2(dy_1^2+dy_2^2). \tag{12}$$

In this metric, at least one of the transverse directions y_i should be compactified, so that the parameter b cannot be changed by rescaling the coordinates. Thus, the only non-vanishing conserved charge is the mass, as is depicted below

$$Q(\tfrac{\partial}{\partial t}) = \tfrac{bV}{8\pi}\ ,\ Q(\tfrac{\partial}{\partial y_1}) = Q(\tfrac{\partial}{\partial y_2}) = 0, \tag{13}$$

where V stands for the two-dimensional transverse volume. Then, the constant $b = \frac{8\pi M}{V}$ is a density of mass, unless both transverse directions be compact. These results are in agreement with the computation of Horowitz and Myers, using a background-dependent method [17].

- **Taub-NUT & Bolt**

Finally, we consider the Euclidean instanton solutions known as Taub-NUT and Taub-Bolt (see, e.g. [18]), whose metrics can be written as [19]

$$ds^2 = \frac{E}{4}\left[\frac{F(r)}{E(\frac{r^2}{l^2}-1)}(d\tau + lE^{\frac{1}{2}}\cos\theta d\phi)^2 + \frac{4(\frac{r^2}{l^2}-1)}{F(r)}dr^2 + (r^2-l^2)(d\theta^2+\sin^2\theta d\phi^2)\right]. \tag{14}$$

Using formula (9), the following results for the mass and angular momentum are obtained for each case:

NUT:

$$Q_N(\tfrac{\partial}{\partial\tau}) = \tfrac{l}{2}E^{\frac{1}{2}}(1-E)\ ,\ Q_N(\tfrac{\partial}{\partial\phi}) = 0 \tag{15}$$

Bolt:

$$Q_B(\tfrac{\partial}{\partial\tau}) = \tfrac{l}{16s}E^{\frac{1}{2}}[Es^4 + (4-6E)s^2 - (3E-4)],$$
$$Q_B(\tfrac{\partial}{\partial\phi}) = 0\ . \tag{16}$$

Using Taub-NUT as the background configuration, Hawking, Hunter and Page computed the mass for the Taub-Bolt geometry. Substracting (15) from (16), one obtains

$$\Delta M = \frac{l}{16s}E^{\frac{1}{2}}[Es^4 + (4-6E)s^2 - (3E-4) + 8(E-1)s],$$

which agrees with [18,20].

From the previous examples it is possible to observe that in all cases the mass and other conserved charges can be computed from the formula (9), without normalizing the Killing vector as is often found in standard approaches [21].

A remarkable feature of formula (9) is that the following general statement can be derived from it:

For any negative constant curvature manifold, the Noether charges associated with diffeomorphisms vanish identically. As a consequence, any locally AdS spacetime which admits a set of global Killing vectors, has all its associated conserved charges identically zero.

This means that any local AdS geometry possessing a time-like Killing vector has vanishing mass. Considering that gravity has non-negative energy [22], the above statement implies that General Relativity has a degenerate vacua: the only way to distinguish them would be through their topological features.

An important problem in this context would be to identify all possible vacua. It is easily seen that suitable vacua can be constructed as quotients of the covering of *AdS* by a discrete group Γ that do not leave timelike or spacelike fixed points, to avoid causal and conical singularities, respectively. In fact, AdS spacetime is clearly not *the unique* possible vacuum, since another choice is obtained from the metric (12) with $b=0$, which corresponds to AdS spacetime with identifications.

Discussion. The Lagrangian in (5) can be understood as follows. The most general 4-form invariant under local Lorentz transformations, constructed with the vielbein, the spin connection, and their exterior derivatives, without using the Hodge dual $(*)$ and even under parity is a linear combination of the standard Einstein-Hilbert Lagrangian with a cosmological term plus the Euler density. Hence, requiring that any ALAdS spacetime be an extremum for the action fixes the weight factor of the Euler density in terms of gravitational and cosmological constant as $-\frac{3\kappa}{4\Lambda}$. This is the key point in order to obtain the charge (9).

Since the Lagrangian is invariant under local Lorentz transformations, substitution of $\delta\omega^{ab} = -D\lambda^{ab}$ in the general expression for the Noether current (7) yields the conserved charge

$$Q(\lambda^{ab}) = \frac{\kappa l^2}{2}\int_{\partial\Sigma}\epsilon_{abcd}\lambda^{ab}\bar{R}^{cd}. \tag{17}$$

This expression is manifestly Lorentz covariant. On the

other hand, in case of diffeomorphism symmetry, $Q(\xi)$ given by (9), transforms under local Lorentz rotations as

$$\delta_\lambda Q(\xi) = -\frac{\kappa l^2}{2}\int_{\partial\Sigma}\epsilon_{abcd}\mathcal{L}_\xi\lambda^{ab}\bar{R}^{cd}. \tag{18}$$

Thus, a sufficient condition to ensure the invariance of $Q(\xi)$ is, as usual, the requirement $\mathcal{L}_\xi\lambda^{ab} = 0$ at the boundary.

Komar's expression for the charge and formula (9) differ substantially although both are deduced from Noether's theorem for diffeomorphisms. In the latter case, the charge comes from an action containing an additional closed form and the entire procedure has been conducted in first order. To make a more explicit comparison, it is convenient to separate the charge (9) as

$$Q(\xi) = K(\xi) + X(\xi) + E(\xi), \tag{19}$$

where $K(\xi) = -\kappa\int_{\partial\Sigma}\nabla^\mu\xi^\nu d\Sigma_{\mu\nu}$ is Komar's integral, $X(\xi) = -\frac{\kappa}{2}\int\limits_{\partial\Sigma}\epsilon_{abcd}\Phi^{ab}e^c e^d$ –with $\Phi^{ab} = e^{a\mu}\mathcal{L}_\xi e^b_\mu$– is a contribution due to the local Lorentz invariance, which does not exist in the second order formalism [23], and finally, $E(\xi) = \frac{\kappa l^2}{2}\int\limits_{\partial\Sigma}\epsilon_{abcd}I_\xi\omega^{ab}R^{cd}$, which results from the Euler density added to the Einstein-Hilbert action. This last term serves two purposes: it eliminates the divergent terms in the explicit evaluation for the solutions, and corrects the normalization factors. From this point of view, this new term regularizes the Noether charges in ALAdS spacetimes. This can be illustrated in the Schwarzschild-AdS solution. In this case $\Phi^{ab} = 0$ and $\xi = \partial_t$, so that the $X(\xi)$ term vanishes and

$$K(\xi) = \tfrac{M}{2} + \lim_{r\to\infty}\tfrac{r^3}{2l^2},\ \ E(\xi) = \tfrac{M}{2} - \lim_{r\to\infty}\tfrac{r^3}{2l^2}. \tag{20}$$

Hence, $Q(\xi) = M$, which is clearly unaffected whether the limit $l\to\infty$ is performed or not, which means that the formula is valid for any value of the cosmological constant, including $\Lambda = 0$. However the construction leading to (19) could not have carried out if $\Lambda = 0$ from the start. Thus, the presence of a cosmological constant can be thought of as a regulator for General Relativity with $\Lambda = 0$.

An interesting issue is finding the generalization of the charge formula (9) for higher dimensional gravity theories which admit a well defined variational principle for ALAdS solutions. Indeed, there is a natural way to extend this result to any even dimension d>4 [24]. For odd dimensions, however, this extension is not straightforward.

Acknowledgments

We are grateful to J. Crisóstomo, J. Gamboa, A. Gomberoff, C. Martínez, F. Méndez, M. Plyushchay, J. Saavedra and C. Teitelboim for many enlightening discussions. We are specially thankful to M. Henneaux for many helpful comments. This work was supported in part through grants 1990189, 1970151, 1980788, 2990018, 3960007, 3960009 and 3990009 from FONDECYT, and grant 27-953/ZI-DICYT (USACH). The institutional support of Fuerza Aérea de Chile, I. Municipalidad de Las Condes, and a group of Chilean companies (AFP Provida, Business Design Associates, CGE, CODELCO, COPEC, Empresas CMPC, GENER S.A., Minera Collahuasi, Minera Escondida, NOVAGAS and XEROX-Chile) is also recognized. R.T. and J.Z. wish to express their gratitude to Marc Henneaux, for kind hospitality in Brussels. Also R.A., R.O. and J.Z. wish to thank to the organizers of the ICTP Conference on Black Hole Physics for hospitality in Trieste this summer.

---

[1] A. Komar, Phys. Rev. **113,** (1959) 934.

[2] J. Lee and R.M. Wald, J. Math Phys. **31**, (1990) 725.

[3] R. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. **117**, (1960) 1595; in *Gravitation: An Introduction to Current Research*, ed. by L. Witten (Wiley,NewYork,1962).

[4] L.F. Abott and S. Deser, Nucl. Phys. **B195**, (1982) 76.

[5] T. Regge and C. Teitelboim, Ann. Phys. (NY) **88**, (1974) 286.

[6] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. **13**, (1996) 1487.

[7] A. Ashtekar and A. Magnon, Class. Quantum Grav. **1** (1984) L39. See also, R.C. Myers, "On the Gravity of the AdS/CFT Correspondence," talk at ITP Conference on Strong Gravitational Fields, June 22-26, 1999.

[8] M.Henneaux and C. Teitelboim, Comm. Math. Phys. **98,** (1985) 391.

[9] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231.

[10] V. Balasubramanian and P. Kraus, *A Stress Tensor for Anti-de Sitter Gravity*, hep-th/9902121.

[11] J.D. Brown and J.W. York, Phys. Rev. **D47**, (1993) 1407.

[12] R. Emparan, C.V. Johnson and R.C. Myers, *Surface Terms as Counterterms in the AdS/CFT Correspondence*, hep-th/9903238.

[13] R.B. Mann, *Misner String Entropy*, hep-th/9903229.

[14] The contraction operator I_ξ acts on a p-form $\alpha_p = \frac{1}{p!}\alpha_{\mu^1\ldots\mu^p}dx^{\mu^1}\cdots dx^{\mu^p}$ as
$$I_\xi\alpha_p := \tfrac{1}{(p-1)!}\xi^\nu\alpha_{\nu\mu^1\ldots\mu^{p-1}}dx^{\mu^1}\cdots dx^{\mu^{p-1}}.$$
The Lie derivative can be written in terms of this operator as $\mathcal{L}_\xi = dI_\xi + I_\xi d$. Therefore, the following useful property holds when it acts over the spin connection, $\mathcal{L}_\xi\omega^{ab} = D(I_\xi\omega^{ab}) + I_\xi R^{ab}$.

[15] B. Carter, Comm. Math. Phys. **17,** (1970) 233.

[16] J. Lemos, Phys.Lett. **B353**, (1995) 46.

[17] G. T. Horowitz and R. C. Myers, Phys. Rev. **D59**, (1999) 026005.

[18] S.W. Hawking, C.J. Hunter and D.N. Page, Phys. Rev. **D59**, (1999) 044033.

[19] $F_N = E\frac{r^4}{l^4} + (4-6E)\frac{r^2}{l^2} + (8E-8)\frac{r}{l} + 4 - 3E$ for the Taub-NUT case and $F_B = E\frac{r^4}{l^4} + (4-6E)\frac{r^2}{l^2} + \left[-Es^3 + (6E-4)s + \frac{3E-4}{s}\right]\frac{r}{l} + 4 - 3E$ for Taub-Bolt, for which $E = (2ks-4)/(3s^2-3)$.

[20] The same expressions for the NUT and Bolt masses have

been found in Ref. [12].
[21] Once normalized, ξ is no longer a Killing vector.
[22] R. Schoen and S. T. Yau, Comm. Math. Phys. **65,** (1979) 45; E. Witten, Comm. Math. Phys. **80**, (1981) 381.
[23] Although it is always possible to choose a frame (e^a) such that $\Phi^{ab} = 0$ in an open neighborhood, the first order formalism could provide interesting cases where there is a global obstruction which makes $X(\xi)$ non-trivial.
[24] R. Aros, M. Contreras, R. Olea, R. Troncoso, J. Zanelli, *Conserved Charges for Even Dimensional AdS Gravity Theories*. Reports, CECS-PHY-99/18, ULB-TH-99/26; e-print: hep-th/9912045.